HP - Small Grids Exploration

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1 Motivation

In the previous session, we identified several sufficient conditions under which a rectangular grid graph with a T-shaped hole admits a Hamiltonian path between vertices s and t, by explicitly constructing such paths. Our construction strategy decomposes the larger grid into smaller subgraphs that are easier to analyze, and then connects Hamiltonian paths in these subgraphs to obtain a Hamiltonian path for the original grid.

During this decomposition process, we observed that most instances can be partitioned into collections of rectangular subgraphs. This observation led us to study Hamiltonian paths in rectangular grid graphs under the restriction that the starting and ending vertices lie on the leftmost or rightmost boundary (though not necessarily on the same side) and either topmost or bottommost rows. We exhaustively explored all possible Hamiltonian paths in these smaller rectangular grids and identified emerging structural patterns governing Hamiltonian paths in an $m \times n$ rectangular grid graph.

2 Definition

In this section, we formally define the class of rectangular grid graphs considered throughout this report. A rectangular grid graph is specified by its dimensions $m \times n$, where m denotes the number of columns (length) and n denotes the number of rows (width).

We impose the restriction that the starting vertex s and ending vertex t lie on either the leftmost or the rightmost vertical edge of the grid, and that they are located on the topmost or bottommost rows. Under these constraints, there are exactly two distinct configurations for the placement of s and t:

1. **Same-edge case.** Both s and t lie on the same vertical edge, either the leftmost edge, or the rightmost edge. Without loss of generality, we assume they are on the leftmost edge, with the starting point s at the topmost row and the ending point t at the bottommost row; that is, $s = (0, 0)$ and $t = (0, n-1)$. The corresponding case where both vertices lie on the rightmost edge can be obtained by a 180 degree rotation of the grid.

2. **Different-edge case.** The vertices s and t lie on opposite vertical edges, again, either the leftmost or the rightmost edge. Without loss of generality, we assume s lies on the leftmost edge and t lies on the rightmost edge, with $s = (0, 0)$ and $t = (m-1, n-1)$. The configuration where these roles are reversed can be obtained by rotating the grid. The only remaining configuration in which s and t lie on different edges is when they are on the same row, which is equivalent to the same-edge case under a 90 degree rotation.

For convenience, we will refer to such graphs as $m \times n$ *same-edge* or *different-edge* rectangular grid graphs, depending on the relative placement of s and t. When we say any $m \times n$ rectangle grid graphs, we assume that the placement for s and t falls into either edge-same or edge-different case.

3 Generating all possible Hamiltonian paths

First, in order to figure out the pattern for all possible Hamiltonian paths in rectangle grid graphs with the starting and the ending points are on the leftmost edge or the rightmost edge, we need to analyze those smaller rectangle grid graph cases. To do so, we implemented the follow algorithm to generate all possible Hamiltonian paths in various sizes of rectangle grid graph with s and t specified.

3.1 Description

In order to keep track of all possible Hamiltonian paths between the starting point s and ending point t, given a $n \times m$ grid, for relatively small grids, we propose the following algorithm to enumerates all Hamiltonian paths on a grid using depth-first search with backtracking. At each step, it extends the current path by trying all adjacent moves (left, right, up, down) that is valid, i.e. inside of the grid and hasn't been visited before. When a path covers all N cells, it is output, and the search backtracks to explore all remaining possibilities.

3.2 Pseudocode

Algorithm 1 Enumerate All Hamiltonian Paths on a Grid

```
Input: Grid G with N cells; starting cell (x0, y0)
Output: All Hamiltonian paths starting from (x0, y0)
Function Main():
    path ← [(x0, y0)]  Explore(path, (x0, y0))

Function Explore(path, current):
    if |path| = N then
        output path  return ; // terminate this branch only
    directions ← {right, left, up, down}
    foreach direction ∈ directions do
        next ← current.direction
        if Valid(next, path) then
            append next to path  Explore(path, next)  remove next from path; // backtrack

Function Valid(cell, path):
    (x, y) ← cell  if (x, y) is outside the grid then
        return false
    if cell ∈ path then
        return false
    return true
```

3.3 Implementation

Considered the large runtime for this algorithm, in order to generate the Hamiltonian paths sets for larger grid, we separate the process of outputting the set of the acceptable Hamiltonian paths and the process of visualizing the paths. A sample output format would be the following:

$$\boxed{start = (0,0); end = (0,1); path = (0,0) \to (1,0) \to (2,0) \to (2,1) \to (1,1) \to (0,1)}$$

4 Splitting strategies for larger rectangle grid graph

To explore the pattern of the Hamiltonian paths for rectangle grid graph with the constraint on the position starting and ending vertices, we need to split the larger grid into smaller gird graphs, which are easier to find the Hamiltonian path, or identify the pattern. We need to find a way to separate the graph such that we can construct the Hamiltonian path for each smaller subgraphs first, and then connecting them together easily to get the Hamiltonian path for the larger graph. We start with exploring those small grid graphs' set that we generated by the implementation above, which include all $m \times n$ same-edge and different-edge grid graphs with $2 \leq m, n \leq 7$.

We will now present some splitting strategies for separating larger rectangle grid graph into smaller subgraphs. We will introduce the **single-crossing lines strategy** that separate the graphs into subgraphs by any (either vertical or horizontal) line that has only been crossed by the Hamiltonian path by once; and also the **separable edge-columns vertices strategy** that splitting the graph into subgraphs by horizontally cutting along those vertices on the edge that separate Hamiltonian path into two completed paths. The separable edge-columns vertices were implemented at first, which is not as complete as the single-crossing lines strategy. Therefore, for future analysis, we will build on the result we got from the implementation of the single-crossing lines strategy.

4.1 Single-crossing lines

To split the larger grid graph into smaller one, we can use the single-crossing lines strategy, which is separating the grid graph either horizontally or vertically by the lines that the Hamiltonian path only cross for once. To find such lines, we can simply trace the Hamiltonian path from the starting point, and record every two consecutive vertices' relative position, which appeals the crossing information for each horizontal/vertical line. Then, we need to iterate through all vertical and horizontal lines to identify those have been crossed only by once. Figure 1 shows a visualized example, where each bold vertical line represents each single-crossing line.

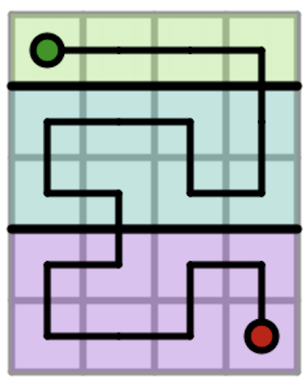


Figure 1: The example for single-crossing lines, emphasized here. Note that each subgraph separated by the single-crossing line contains a complete Hamiltonian path, which can be easily connected to others to get the Hamiltonian path for the whole grid graphs.

4.1.1 Pseudocode

Algorithm 2 Detecting Separable Rows and Columns from a Hamiltonian Path

```
Input: An m × n grid G; a Hamiltonian path H on G
Output: Sets C of separable columns and R of separable rows
Function Main():
    (vertical, horizontal) ← BuildEdgeUsage(H)  (C, R) ← ExtractSeparableCuts(m,n,vertical,horizontal)
    return (C, R)

Function BuildEdgeUsage(H):
    Initialize vertical[0 ... m − 2][0 ... n − 1] ← 0  Initialize horizontal[0 ... m − 1][0 ... n − 2] ← 0
    foreach consecutive vertices (x, y) → (x′, y′) in H do
        if x′ = x + 1 or y′ = y + 1 then
            vertical[x][y] ← vertical[x][y] + 1
        else if x′ = x − 1 then
            vertical[x′][y] ← vertical[x′][y] + 1
        else if y′ = y − 1 then
            horizontal[x][y′] ← horizontal[x][y′] + 1
    return (vertical, horizontal)

Function ExtractSeparableCuts(m, n, vertical, horizontal):
    Initialize C ← ∅  Initialize R ← ∅
    // Identify separable columns
    for x = 0 to m − 2 do
        count ← 0  for y = 0 to n − 1 do
            if vertical[x][y] = 1 then
                count ← count + 1
        if count = 1 then
            C ← C ∪ {x}
    // Identify separable rows
    for y = 0 to n − 2 do
        count ← 0  for x = 0 to m − 1 do
            if horizontal[x][y] = 1 then
                count ← count + 1
        if count = 1 then
            R ← R ∪ {y}
    return (C, R)
```

4.2 Separable edge-columns vertices

We have tried another way of splitting the larger grid graph too, which turns out was included in the single-crossing line case that we decided to use. The strategy is separate horizontally along separable vertex on the edge columns. This strategy search for the complete horizontal line and the complete vertical line first. Then, among all cases that don't include neither a complete horizontal line nor a complete vertical line, we will try to identify those separable vertices on the edge columns and then separate the graph horizontally by those separable vertices into several subgraphs. A vertex v on the edge column is considered as a separable vertex if every vertex that was reached before that v by the Hamiltonian path, has $y-$coordinate at most that of v. For every separable vertex on the edge, we can separate the larger graphs into two subgraphs by the horizontal line right below the separable vertex, as shown in the Figure 2.

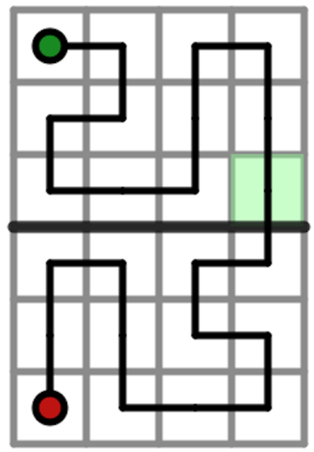


Figure 2: The example for a separable vertex, highlighted by green. Where the upper subgraph has a complete Hamiltonian path between s and the separable vertex; so does the lower subgraph between the vertex below the separable vertex and t. Note: the green grid represents s, and the red grid represents t.

Note that for any separable edge-columns vertex, since, by definition, every vertex reached by the Hamiltonian path before itself is above or on the same level ($y-$coordinate value) of it, and every vertex reached after it is below, the horizontal line below it is also a single-crossing line. Thus, this strategy is covered under single-crossing lines. We will use single-crossing line for future analysis.

4.2.1 Pseudocode

The above separable vertices on the edge columns strategy has been implemented as below:

Algorithm 3 Finding the separable vertices

Input: Grid G, starting and ending vertices s and t, Hamiltonian path H
Output: All separable vertices on the edge

```
Function Separable_vertices(G, s, t, H):
    result = [];
    // a = (x, y), with x = 0 or x = n − 1
    foreach a on the leftmost/rightmost edge do
        // for every point above or on a
        points = {every point with y−value ≤ y}
        foreach grid_covered ∈ H do
            // stop when we reach the testing vertex a
            if gird_covered == a then
                break;
            if gird_covered ∈ points then
                remove grid_covered from points;
            else
                // if any vertex reached by the Hamiltonian path
                // before reaching a are below a
                // go to the next vertex to test
                continue;
        if points.isEmpty() then
            // if all vertices with y−value leqy are visited by H
            // and none of the vertices below a are visited
            // both before it reaches a
            // a is a separable vertex on the edge columns
            result.append(a);
```

5 Number of Hamiltonian paths

We recorded the number of Hamiltonian paths for each $m \times n$ same-edge/different-edge rectangle gird graph. We can found out some the pattern of the existence of the Hamiltonian path and the number of the Hamiltonian paths along with the increasing of the size of the rectangle grid graph.

m	n	$s\&t$	number of Hamiltonian paths	m	n	$s\&t$	number of Hamiltonian paths
2	2	same	1	2	2	diff	0
2	3	same	0	2	3	diff	1
2	4	same	1	2	4	diff	0
2	5	same	0	2	5	diff	1
2	6	same	1	2	6	diff	0
2	7	same	0	2	7	diff	1
3	2	same	1	3	2	diff	1
3	3	same	2	3	3	diff	2
3	4	same	4	3	4	diff	4
3	5	same	8	3	5	diff	8
3	6	same	16	3	6	diff	16
3	7	same	32	3	7	diff	32
4	2	same	1	4	2	diff	0
4	3	same	0	4	3	diff	4
4	4	same	8	4	4	diff	0
4	5	same	0	4	5	diff	20
4	6	same	47	4	6	diff	0
4	7	same	0	4	7	diff	111
5	2	same	1	5	2	diff	1
5	3	same	4	5	3	diff	8
5	4	same	23	5	4	diff	20
5	5	same	86	5	5	diff	104
5	6	same	397	5	6	diff	378
5	7	same	1584	5	7	diff	1670
6	2	same	1	6	2	diff	0
6	3	same	0	6	3	diff	16
6	4	same	55	6	4	diff	0
6	5	same	0	6	5	diff	378
6	6	same	1770	6	6	diff	0
6	7	same	0	6	7	diff	10204
7	2	same	1	7	2	diff	1
7	3	same	8	7	3	diff	32
7	4	same	144	7	4	diff	111
7	5	same	948	7	5	diff	1670
7	6	same	11658	7	6	diff	10204
7	7	same	88418	7	7	diff	111712

Table 1: Number of Hamiltonian paths for $m \times n$ same-edge/different-edge rectangles with $2 \leq m, n \leq 7$

5.1 The existence of the Hamiltonian path

From Table 1, we observe that any $m \times n$ *edge-same* rectangle with m even and n odd admits no Hamiltonian path. Likewise, any $m \times n$ *edge-different* rectangle with both m and n even has no Hamiltonian path; this case can be mapped to a configuration that has been analyzed previously. In *Hamilton Paths in Grid Graphs*, [Itai et al.(1982)] showed that for an even-sized rectangle grid graph, when $m \times n$ is even, a Hamiltonian path exists only if the starting vertex s and the ending vertex t have different parity. We show that this parity obstruction holds in the same way for all rectangular grid graphs under our edge-position constraints. Specifically, we claim the following theorem:

Theorem 5.1. *For any $m \times n$ rectangle grid graph, there is no Hamiltonian path between s and t when*

1. *m is even, n is odd, and s and t fall into the* edge-same *case; or*
2. *both m and n are even, and s and t fall into the* edge-different *case.*

We prove this by showing that in both situations, the configurations force s and t to have the same parity, thereby violating the necessary parity condition for the existence of a Hamiltonian path.

Proof. Recall that color parity is determined by the relative positions of vertices in the grid: any two adjacent vertices have different colors (black and white), while vertices at an even graph distance share the same color parity. In particular, a vertex that is two units away from another vertex has the same color parity. For two vertices $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, this parity relationship can be characterized by the parity of their Manhattan distance $|x_1 - x_2| + |y_1 - y_2|$. By observation, when the Manhattan distance between two vertices is even, the vertices have the same color parity; when it is odd, they have different color parity.

When m is even, n is odd, and s and t fall into the *edge-same* case, $s = (0, 0)$ and $t = (0, n-1)$. The Manhattan distance between s and t is $|0 - 0| + |(n-1) - 0| = n - 1$, which is even since n is odd. Hence, s and t have the same color parity. Similarly, when m is even, n is even, and s and t fall into the *edge-different* case, $s = (0, 0)$ and $t = (m-1, n-1)$. The Manhattan distance between s and t is $|(m-1) - 0| + |(n-1) - 0| = m + n - 2$, which is even. Thus, s and t also have the same color parity in this case.

By the theorem of Itai et al. cited above, the existence of a Hamiltonian path in an even-sized grid graph requires the endpoints to have different color parity. Therefore, in both cases described above (both of them are even-sized since m is even for all), no Hamiltonian path exists. This completes the proof that there is no Hamiltonian path between *edge-same* vertices when m is even and n is odd, nor between *edge-different* vertices when both m and n are even.

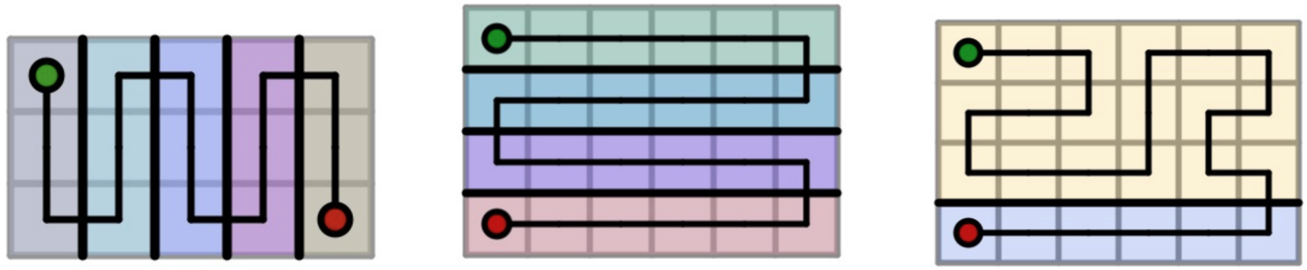
□

5.2 The of Hamiltonian paths for $3 \times n$ rectangle grid graphs

In this section, we will propose two different strategies to construct the Hamiltonian path on $3 \times n$ rectangle grid graphs. We will first define some necessary terms, and explain and prove our observation for the number of Hamiltonian paths for $3 \times n$ rectangle grid graphs by showing that every path we constructed under the strategies proposed are valid, and we cannot build other Hamiltonian path that is not built by those two strategies.

5.2.1 Preliminary definitions

Definition 5.2 (Fully Vertical/Horizontal Hamiltonian Paths)**.** A fully vertical/horizontal Hamiltonian path is a path that constructed by the following way: starting from the starting point s, keep going vertically/horizontally, and turn only when necessary, i.e. make a unit U-turn when we can't keep going in the same direction. Examples of fully vertical, fully horizontal, and a counterexample are shown in Figure 3.



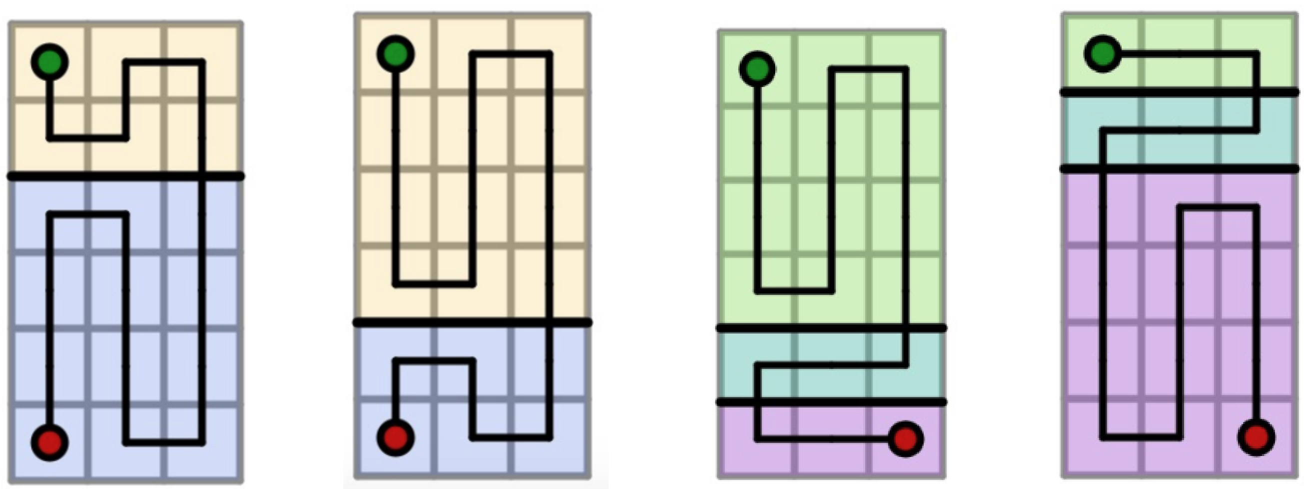
(a) Fully vertical (b) Fully horizontal (c) A counterexample

Figure 3

Lemma 5.3. *For any $3 \times n$ edge-different rectangle grid graph, there exist a fully vertical Hamiltonian path, as shown in Figure 3a; and for any $3 \times n$ edge-same rectangle grid graph, there exsit a fully horizontal Hamiltonian path, as shown in Figure 3b.*

Definition 5.4 (Vertical Intercepts)**.** Given two vertically adjacent vertices v_1 and v_2 in the rectangle grid graph, i.e. $v_1 = (x_1, y_1)$ and $v_2 = (x_1, y_1 - 1)$, and a Hamiltonian path p, we say $v_1 - v_2$ is a vertical intercept of p if v_1 and v_2 are consecutive along this Hamiltonian path.

Definition 5.5 (Non Horizontally Separable Hamiltonian Paths)**.** We call a Hamiltonian path for the $m \times n$ rectangle grid graph is non horizontally separable if there is no such a horizontal line can separate the Hamiltonian path into two sub-paths that both are completed rectangular paths. In another word, for a Hamiltonian path to be non horizontally separable, at each row, there has to exist more than one vertical intercepts. For example, a fully vertical Hamiltonian path, as shown in Figure 3a, is not horizontally separable, but any Hamiltonian path that contains a $m - 1$ length of horizontal part, as shown in Figure 3c, is considered as a horizontally separable path.



(a) HP for 3×6 edge-same (b) HP for 3×6 edge-same (c) HP for 3×6 edge-diff (d) HP for 3×6 edge-diff

Figure 4: Examples for distinct Hamiltonian paths

Lemma 5.6. *For a $3 \times n$ rectangle grid graph, the only possible non horizontally separable Hamiltonian path is fully vertical. I'm pretty sure this is true, also checked the paths output from the program, but not sure how to prove.*

Definition 5.7 (Endpoint-Swapping Symmetries)**.** Let G be a rectangular grid graph with designated vertices s and t.

We define the following graph symmetries for different $s-t$ positioning cases, each of which exchanges the roles of s and t:

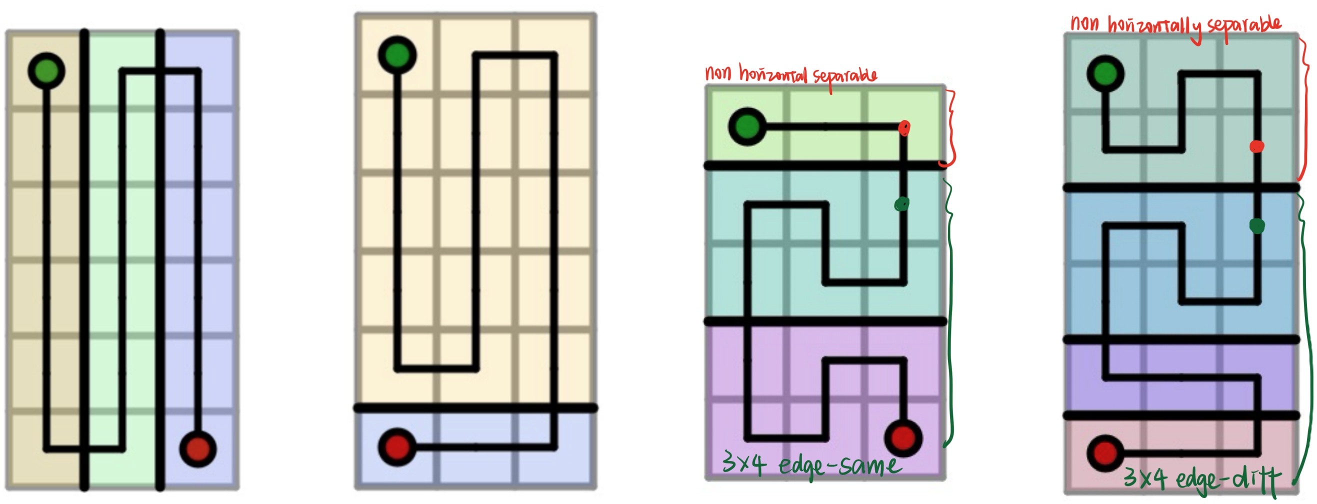
1. **Edge-same case (horizontal reflection).** The *horizontal reflection*, denoted by ϕ_{ref}, is the graph automorphism obtained by reflecting G across the horizontal axis that separates the two regions of the graph. This reflection preserves adjacency and maps the endpoint s to t and t to s. An example of horizontal reflection is shown in Figure 4a and 4b.

2. **Edge-different case (180° rotation).** The 180° *rotation*, denoted by ϕ_{rot}, is the graph automorphism obtained by rotating G by 180° about its center. This rotation preserves adjacency and exchanges the endpoints s and t. An example of 180° rotation is shown in Figure 4c and Figure 4d.

Definition 5.8 (Distinct Hamiltonian Paths)**.** Let P and Q be Hamiltonian paths in G with endpoints (s, t). We say that P and Q are *distinct Hamiltonian paths* if they are not identical as sequences of vertices in G.

In particular, even if there exists an endpoint-swapping symmetry $\phi \in \{\phi_{\mathrm{ref}}, \phi_{\mathrm{rot}}\}$ such that $\phi(P) = Q$, the two paths P and Q are still considered distinct. Figure 4a and Figure 4b are considered as two distinct Hamiltonian paths, and so do Figure 4c and Figure 4d. That is, Hamiltonian paths are counted without quotienting by endpoint-swapping symmetries of the graph.

5.2.2 The strategies for constructing Hamiltonian paths



(a) strategy 1: edge-diff (b) strategy 1: edge-same (c) strategy 2: edge-diff (d) strategy 2: edge-same

Figure 5: Examples for different strategies under different scenarios

For any $3 \times n$ rectangle grid graphs with s and t being either edge-same or edge-different, we can construct the Hamiltonian paths by following different strategies we proposed below, each with a different number of possibilities, i.e. different numbers of possible Hamiltonian paths. Note that the number of Hamiltonian paths for a $3 \times n$ rectangle grid graph is denoted by $HP_{3 \times n_{\mathrm{same/diff}}}$:

1. We can construct one Hamiltonian path by connecting s and t fully vertically for the edge-difference case (as Figure 5a), and connecting s and t by combining the fully vertical Hamiltonian path for the $3 \times (n-1)$ edge-same case with the horizontal line covering the last row (as Figure 5b);

2. For a $3 \times n$ edge-different rectangle grid graph, we can construct $HP_{3 \times (n-n')_{\mathrm{same}}}$ Hamiltonian paths by connecting the upper part, the non horizontally separable Hamiltonian path of the $3 \times n'$ edge-different grid graph, to the lower part, the Hamiltonian path of $3 \times (n-n')$ edge-same rectangle grid

graph, which forms the Hamiltonian paths for $3 \times n$ edge-different rectangle grid graph, as shown in Figure 5c. Thus, for every $1 \leq n' \leq n-2$, we can have $HP_{3\times(n-n')_{\text{same}}}$ Hamiltonian paths, thus $\sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{\text{same}}}$ in total.

Similarly, for the edge-same case, we can combine the non horizontally separable Hamiltonian path of $3\times n'$ edge-different grid graph (the upper part) to each possible Hamiltonian path for $3\times(n-n')$ edge-different, the lower part (as shown in Figure 5d), which gives us $\sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{\text{diff}}}$ many Hamiltonian paths constructed.

5.2.3 The number of Hamiltonian paths

By observation, we found out that the number of Hamiltonian paths for $3 \times n$ rectangle grid graphs can be express by the number of Hamiltonian paths for smaller rectangle grid graphs, since they were constructed by combining each possible Hamiltonian path for some smaller subgraphs and a corresponding non horizontal separable subgraph.

Theorem 5.9. *For any $3 \times n$ rectangle grid graph, the number of possible Hamiltonian paths can be expressed as following:*

$$HP_{3\times n_{same}} = 1 + \sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{diff}} \quad \textit{for edge-same case;}$$
$$HP_{3\times n_{diff}} = 1 + \sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{same}} \quad \textit{for edge-different case.}$$

Proof. We will now prove this theorem by showing the following two hold:

1. $HP_{3\times n_{\text{same/diff}}} \geq 1 + \sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{\text{diff/same}}}$, which we will prove this by showing the validity of both strategies we proposed above.
2. $HP_{3\times n_{\text{same/diff}}} \leq 1 + \sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{\text{diff/same}}}$, which we will prove by showing that we cannot build any other Hamiltonian path that is not built by those two strategies, i.e. the completeness.

1. The validity of both strategies:

We will firstly show the validity of strategy 1. By lemma 5.3, there exists a fully vertical Hamiltonian path for any $3 \times n$ edge-different rectangle grid graph. Thus, strategy 1 is always valid for $3 \times n$ edge-different case. For any $3 \times n$ edge-same case, let $t' = (2, n-2)$, i.e. the vertex on top of the bottom-right vertex $s' = (2, n-1)$. Inside of the upper $3 \times (n-1)$ rectangle grid graph, s and t' form the edge-different case. Then, we can always construct the fully vertical Hamiltonian path between s and t' in the $3\times(n-1)$ rectangle subgraph. Then, simply connecting the horizontal line on the bottom row, i.e. between s' and t, to the vertex t' could give us a valid Hamiltonian path since all vertices are covered and none of the vertex has been covered more than once.

For strategy 2, in order to show we can have $\sum_{1\leq n'\leq n-2} HP_{3\times(n-n')_{\text{diff/same}}}$ many valid Hamiltonian paths, we need to show that each path we created is valid AND there is no replicated paths among all paths constructed by strategy 2.

By lemma 5.3, since for each edge-different $3 \times n'$ rectangle grid graph, there is a fully vertical Hamiltonian path, we can always combine such a Hamiltonian path to the lower part of the rectangle grid graph (with sized $3\times(n-n')$). And since by above, there is at least one Hamiltonian path that we can constructed for any rectangle grid graph with either edge-same or edge-different and $m = 3$. Hence, the number of Hamiltonian paths we can build for every $3\times n$ grid graph is at least equivalent to the number of possible Hamiltonian paths of the lower part, i.e. the $3\times(n-n')$ edge-same/different rectangle grid graph, if there is no replicated path. Because the strategy 2 build the Hamiltonian paths by only combining the fully vertical upper part with each possible lower part's Hamiltonian paths together, by induction on the height of the rectangle grid graph and the definition of Distinct Hamiltonian paths (Definition 5.8), all of them are distinct. Thus, we have showed that the validity for all $\sum_{1 \le n' \le n-2} HP_{3\times(n-n')_{\text{diff/same}}}$ Hamiltonian paths constructed by strategy 2.

2. The completeness of two strategies: For a Hamiltonian path, it is either horizontally separable or non horizontally separable. Thus, we will prove the completeness of two strategies by splitting all possible Hamiltonian paths for the $3 \times n$ rectangle grid graphs into two cases: one is non horizontally separable, one is horizontally separable. We will analysis both cases for edge-different and edge-same individually.

1. **Edge-different**: By Lemma 5.6, for a $3 \times n$ edge-different rectangle grid graph, the only possible non horizontally separable Hamiltonian path is the fully vertical path. Those two strategies would include all possible non horizontally separable path since strategy 1 will construct the fully vertical one.

   For the case when the Hamiltonian paths are horizontally separable, each of them could be separated along one of $n-1$ rows' boundary. We will prove that the strategy 2 constructs complete set of the Hamiltonian paths by showing the completeness for each possible rows' boundary separation, by induction. The base case is when $n = 3$: the strategy 2 compute all possible Hamiltonian paths for the separation along the boundary of row 1 and row 2, and the boundary of row 2 and row 3. Then, assume that for any $3 \le n' \le n-1$, strategy 2 can successfully construct all possible horizontally separable Hamiltonian path.

   For $3 \times n$ rectangle grid graph, according to strategy 2, we will construct the Hamiltonian path by connecting each possible upper-lower part separation: connecting the 3×1 edge-different fully vertical Hamiltonian path with all possible $3 \times (n-1)$ edge-same Hamiltonian paths, ..., and connecting the $3 \times (n-2)$ edge-same fully vertical Hamiltonian path with the possible Hamiltonian path for 3×2 edge-same Hamiltonian path. Note that we do not need to include the separation that the upper part is $3 \times (n-1)$ edge-different and 3×1 edge-same because 3×1 edge-same is not a valid Hamiltonian path problem in grid graph setting, thus there is no valid Hamiltonian path under this separation.

   For each possible separation, since the upper part only has one possible path, i.e. fully vertical, and the lower part's all possible Hamiltonian paths could be found by inductive hypothesis, we can find all possible Hamiltonian paths that can be separated by its corresponding upper-lower part's separation. Since we include all possible upper-lower part separations, which by defined is all possible horizontal separation, we can find all possible Hamiltonian paths that is horizontally separated.

2. **Edge-same**: By Lemma 5.6, since the only possible non horizontally separable Hamiltonian path is fully vertical, and we know that an edge-same $3 \times n$ rectangle grid graph cannot have the fully vertical Hamiltonian path (the right two columns will not be covered), there is no non horizontally separable Hamiltonian path for the edge-same situations.

   Because of the similar reason mentioned above, the strategy 2 can construct all possible Hamiltonian paths that is horizontally separable. The only difference is by strategy 2, they also consider the separation with the upper part as a $3 \times n - 1$ edge-different fully vertical rectangle and the lower part as the 3×1 edge-different Hamiltonian path, since the lower part would be a simple straight line, which could be combined to a valid Hamiltonian path.

Hence, we have shown the equality for both edge-same and edge-different cases by proving the validity of both strategies and the completeness of two strategies. □

References

[Itai et al.(1982)] Alon Itai, Christos Papadimitriou, and Jayme Szwarcfiter. 1982. Hamilton Paths in Grid Graphs. *SIAM J. Comput.* 11 (11 1982), 676–686. https://doi.org/10.1137/0211056doi:10.1137/0211056