Hamilton Paths for Rectangle Grid Graphs with Triangle Holes

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Additional Key Words and Phrases: Hamilton paths, grid graph

1 Case Analysis

In this part, we will show that for a rectangle grid graph with a T-shape hole that has a Hamilton Path from s to t, we can construct a Hamilton Path in it by constructing the Hamilton Cycle. We will separate the given rectangle grid graph with a T-shape hole into several subgraph that we can better analysis on. Then, we construct the Hamilton Cycle/Path on each subgraphs, which we proved that such a Hamilton Cycle/Path exists, then simply connect each subgraph's Hamilton Cycle/Path together.

We define the following subgraphs: G_1 (bottom), consisting of the following vertices $V(G_1) = \{(x, y) \mid 0 \leq x \leq m-1, h_0 + a \leq y \leq n-1\}$; G_2 (upper middle), $V(G_2) = \{(x, y) \mid d_0 - 1 \leq x \leq m - c_0, n - a \leq y \leq n-1\}$; G_3, the upper left part, with $V(G_3) = \{(x, y) \mid x \leq d_0 - 1, h_1 + b \leq y \leq n-1\}$; G_4, the upper right part, with $V(G_4) = \{(x, y) \mid m - c_0 \leq x, h_1 + b \leq y \leq n-1\}$; G_5, a subgraph of G_1, with all vertices below the hole, i.e. $V(G_5) = \{(x, y) \mid 0 \leq x \leq m-1, 0 \leq y \leq b-1\}$. And, we define the following vertices: the starting vertex of G_1, $s' = (0, h_1 + b)$, and the ending vertex, $t' = (m-1, h_1 + b)$; the starting vertex of G_2, $s'' = (d_0 - 1, n-1)$, and the ending vertex of G_2, $t'' = (m - c_0, n-1)$; the starting vertex $u = (0, b-1)$ of G_5 and the ending vertex $v = (m-1, b-1)$.

Lemma 1.1. *For any rectangle $R(m, n)$, if n is even, the Hamilton Path can be created by:*

(1) *if s is the top left vertex and t is bottom left vertex, then $s = (0, n-1)$, $t = (0, 0)$, and m is even, we can construct a Hamiltonian path by using strategy 1, go horizontally, then turn when necessary*
(2) *if $s = (0, n-1)$ and $t = (m-1, 0)$, we can construct one by using strategy 2, go vertically, then turn when necessary*

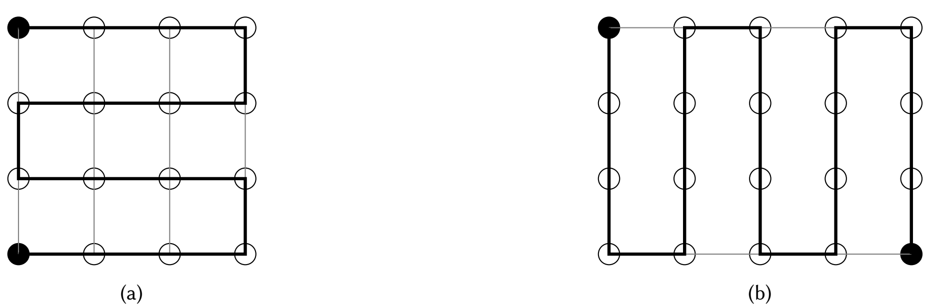

Fig. 1

Proof. To-do ◻

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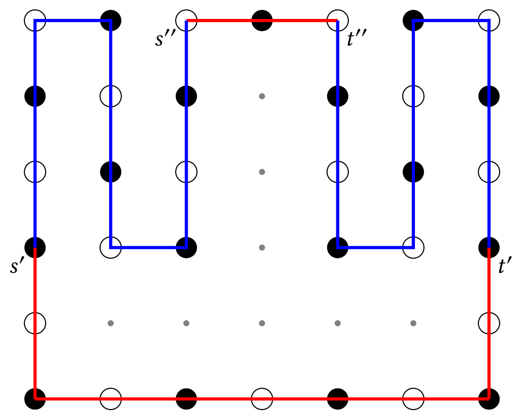
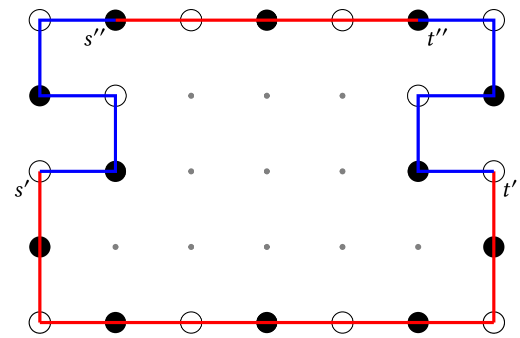
1.1 $a = b = c_1 = d_1 = 1$

Recall that for constructing the Hamilton Path for the 2-rectangle, we have two strategies. Strategy 1 is to turn while going straight (don't turn until you reach the end), which is equivalent to going horizontally, then turn when necessary. And strategy 2 is to go straight while turning (keep turning), which is equivalent to going vertically, then turning when necessary.

Given a rectangular grid graph with a T-shape hole inside with $a = b = c_1 = d_1 = 1$, we propose ways to construct the Hamilton path.



(a) $m = 7, n = 6, a = b = 1, d_0 = c_0 = 3, h_0 = 3, w_0 = 1, d_1 = c_1 = 1, h_1 = 1, w_1 = 5$



(b) $m = 7, n = 5, a = b = 1, d_0 = c_0 = 2, h_0 = 2, w_0 = 3, d_1 = c_1 = 1, h_1 = 1, w_1 = 5$

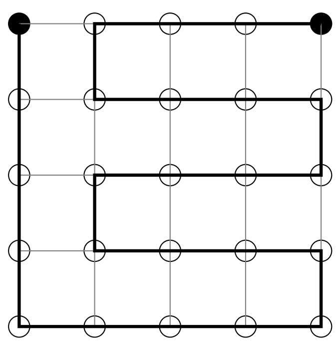
Fig. 2. $a = b = c_1 = d_1 = 1$

1.1.1 $c_0 = d_0 =$ *odd and* $h_0 > 1$. , then we can construct the grid graph as shown in fig 2a. The red line shows the only possible path to cover that grid defined by the conditions; the blue line shows the path that we constructed to cover that grid. For this case, we can construct the Hamilton Path by $|c_0 - 1|$ vertical turns that connect the starting s' vertex of bottom red line area G_1, and the starting vertex s'' of the upper middle part G_2. We can connect them by starting from s'), and keep going until we reach the end, then turn. We can simply keep doing this until we reach s''. Let's consider this rectangle area G_3 with s' and s'', which fall into the first case of the lemma 1.1. Hence, there is a Hamilton path between s' and s'' in G_3. We can construct the right area G_4 similarly. Hence, this Hamilton Path construction works for all grid graphs that fall into this condition.

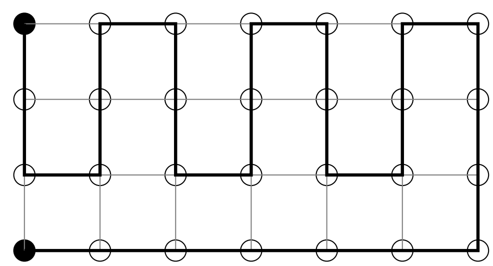
1.1.2 $c_0 = d_0 =$ *even and* h_0 *is even.* , then similarly, we can construct the grid graph as shown in fig 2b. For the left part, we need to connect the starting vertex s' of the bottom part G_1 and the starting vertex s'' of the upper middle part. For the rectangle subgraph G_3, since c_0 is even, and h_0 is even, $a = 1$, which gives us a 2-rectangle with odd height by rotating 90 degrees. Given the position of s' and s'' according to G_3, this fall into the first case of the lemma 1.1. Hence, we can construct the Hamilton path for G_3 and also G_4, shown in Fig. 2b. Thus, this Hamilton Path construction works for all grid graphs that fall into this condition.

Lemma 1.2. *For the rectangle grid graph with $n = \text{ odd } > 1$, we can construct a Hamilton Path between $s = (0, n-1)$ and $t = (m-1, n-1)$, that (1) go from s down to $(0,0)$, then (2) go horizontally until reach the end, make a U-turn, then go to the other direction horizontally, repeat step (2) until reach t.*

Proof. By observation and also mentioned in previous lemma (not sure which one but will check), we found out that even number of turns (odd number of rows) will result in the ending vertex in the other side of rectangle compared to the starting vertex. First, connecting the part on the left with $x = 0$ vertically. In this condition, after step (1), for the subgraph with $x \geq 1$, for sure every gird graph with more than one vertex can achieve, the starting vertex is $(1, 0)$, since we know n is odd, then it ends at vertex in the other side, with x value as $m - 1$ and $y = n - 1$, which is what we want. Hence, by following those two steps, we can successfully construct the Hamilton Path that we want. We have two examples that demonstrate this construction shown in Fig. 3, where in Fig. 3a shows when m is odd, and in Fig. 3b (rotate clockwise 90 degrees), shows when m is even. □



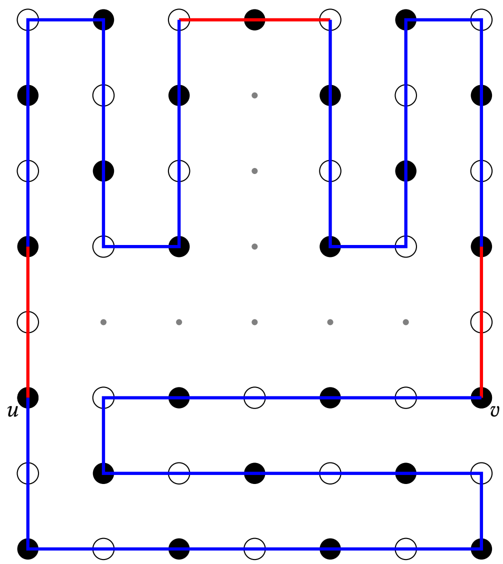
(a)



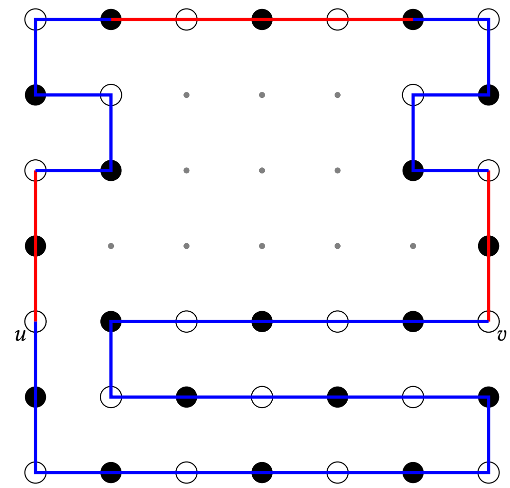
(b)

Fig. 3

1.2 $a = c_1 = d_1 = 1$ and $b =$ odd > 1



(a) $m = 7, n = 8, a = 1, b = 3, d_0 = c_0 = 3, h_0 = 3, w_0 = 1, d_1 = c_1 = 1, h_1 = 1, w_1 = 5$



(b) $m = 7, n = 7, a = 1, b = 3, d_0 = c_0 = 2, h_0 = 2, w_0 = 3, d_1 = c_1 = 1, h_1 = 1, w_1 = 5$

Fig. 4. $a = c_1 = d_1$ and $b =$ odd > 1

Given a rectangular grid graph with a T-shape hole inside with $a = c_0 = d_0 = 1$ and $b =$ odd > 1, we propose ways to construct the Hamilton Path. With the other conditions holding the same, the only thing that we need to consider compared to the previous case is the bottom subgraph, i.e. the Hamilton Path between u and v for the subgraph G_5. As shown in Fig. 4, we just need to go to the bottom, starting from u, go vertically until reach the edge, then switch to the other horizontal direction, and follow the strategy that keep going straight and then turn when necessary. Since the number of lines is odd ($b =$ odd) and u and v are on the left upper most and right upper most vertex, according to lemma 1.2, we can construct the Hamilton path as we defined. Thus, this can be simply separated into two sub cases:

1.2.1 if $c_0 = d_0 =$ odd and $h_0 > 1$. This is demonstrated in Fig. 4a and

1.2.2 if $c_0 = d_0 =$ even and h_0 is even. This is shown in Fig. 4b.

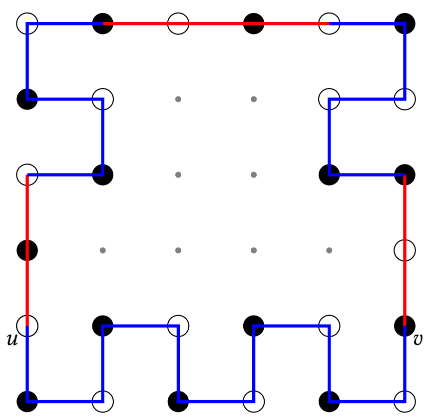
1.3 $a = c_1 = d_1 = 1$, $b =$ even and $m =$ even

Given a rectangular grid graph with a T-shape hole inside with $a = c_1 = d_1 = 1$, $b =$ even and $m =$ even, we propose the following ways to construct the Hamilton Path. Similarly, we can construct the upper parts, i.e. every vertices with $y \geq b$, with two different cases:

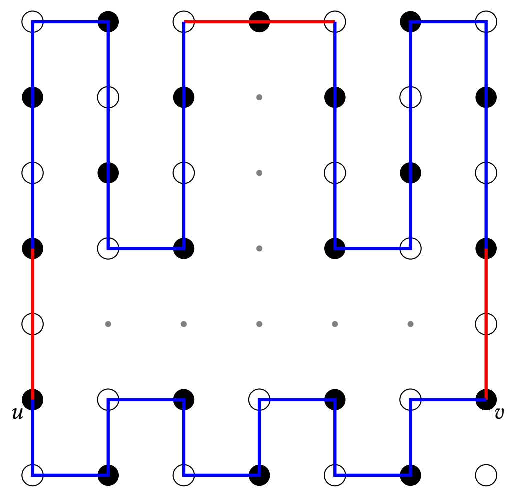
(1) $c_0 = d_0 =$ odd and, $h_0 > 1$

(2) $c_0 = d_0 =$ even and h_0 is even, shown in Fig. 5a

Then, let's check the lower part, G_5, as defined in previous case. According to Fig. 5, by the lemma 1.1 about constructing the Hamilton Path for even rectangle between u and v, where they have the same y value and are at two opposite side of the graph (leftmost and rightmost). Then, with $m =$ even, and the position of u and v, this fall into the second case of lemma 1.1. We can construct this part by going horizontally, then turn when necessary. This is valid for any $b =$ even . By the lemma about 2−rectangle, we can also show that if $m =$ odd and $b = 2$, there is no Hamilton Path can be constructed for such a grid graph, counterexample shown in Fig. 5b.



(a) $m = 6, n = 6, a = 1, b = 2, d_0 = c_0 = 2, h_0 = 2, w_0 = 2, d_1 = c_1 = 1, h_1 = 1, w_1 = 4$



(b) $m = 7, n = 7, a = 1, b = 2, d_0 = c_0 = 3, h_0 = 3, w_0 = 1, d_1 = c_1 = 1, h_1 = 1, w_1 = 5$

Fig. 5. $a = c_1 = d_1 = 1$, $b =$ even and $m =$ even

1.4 $d_1 = a = 1$, b =odd, h_0 and h_1 are even

- non symmetric case

Given a rectangle grid graph with a T-shape hole inside with $d_0 = a = 1$, $b =$ odd , and h_0 and h_1 are even, we propose the following way to construct the Hamilton Path. Let's construct the Hamilton Path for all parts similarly as case **1.2** other than the right subgraph, G_4 as we defined before. Similarly those will need to be separated into two sub cases:

(1) if $c_0 = d_0 =$ odd and $h_0 > 1$
(2) if $c_0 = d_0 =$ even and h_0 is even

Then, for the subgraph G_4, the part on the right, we can connect the Hamilton Path by the paths within two rectangles with even heights, denoted as G_6 with $V(G_6) = \{b \le y \le b + h_1 - 1, m - c_1 \le x\}$, and G_7 with $V(G_7) = \{b + h_1 \le y \le n - a - 1, m - c_0 \le x\}$. Also denote the vertex on the right edge of the grid graph, and the smallest y value in G_5 as u_0, the one with same x value but the largest y value in G_5 as v_0; similarly u_0 and v_0 for G_6, as shown in Fig. 6. By lemma 1.1, the first case, we can show that such a Hamilton Path can always be constructed by designed. Then, we can simply connect v_0 to $(m-1, n-1)$, connect u_0 to v_1, and connect u_1 to $(m-1, b+h_1-1)$ Hence, we can construct the Hamilton Path for such a rectangle grid graph with the T-shaped hole. Note that this case can also be expanded to larger cases such that all h_i are even, with other conditions remain the same.

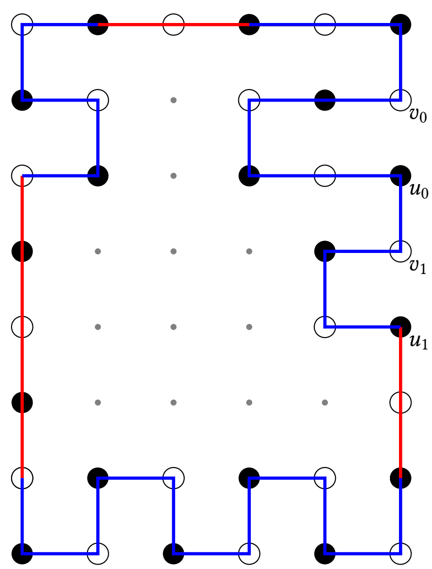


Fig. 6. $d_0 = a = 1$, b =odd, h_0 and h_1 are even

1.5 $d_0 = c_0 = 1, a = \textbf{ odd } > 1,$ **and** $b = \textbf{ odd } > 1$

Given a rectangle grid graph with a T-shaped hole inside with $d_0 = c_0 = 1, a = \text{ odd } > 1,$ and $b = \text{ odd } > 1$, we propose the following ways to construct the Hamilton Path. This will look pretty similar to case **1.2**, where the difference is in the upper part of the grid graph.

To construct the Hamilton Path for this grid graph, first let's just construct the Hamilton Path exclude the subgraphs $G_8 = \{d_1 \leq x \leq d_0 - 1, b + h_1 \leq y \leq n - a - 1\}$ and $G_9 = \{m - c_0 \leq x \leq m - c_1 - 1, b + h_1 \leq y \leq n - a - 1\}$. Those two subgraphs are the one the left and right side of the upper part of the T-shape hole. When we are considering this graph without G_8 and G_9, the graph's upper part, i.e. any vertices with y value greater than (not equal to) $n - a$, will fall into the same case as the lower part, i.e. any vertices with y value smaller than (not equal to) $b - 1$, which is a rectangle grid graph with odd height. Then, we can construct the Hamilton Path for this grid graph exclude subgraphs G_8 and G_9 by handling the upper part and lower part with similar approach as the way we handle the lower part for case 1.2.

Then, let's define the following two horizontal edges that each connects two vertices: e_1 as the edge right above the subgraph G_8, connecting $e_{1_s} = (d_0, n - a)$ and $e_{1_t} = (d_1 - 1, n - a)$; e_2 as the edge right above the subgraph G_9, connecting $e_{2_s} = (m - c_1, n - a)$ and $e_{2_t} = (m - m - c_0 - 1, n - a)$. Break those two edges, then connect it by analyzing the rectangle right below those two edges (included), subgraph $G_8 \cap e_1$ and $G_9 \cap e_2$. They will the rectangles with width $d_0 - d_1$ (G_8) and $c_0 - c_1$ (G_9), and the height as $h_0 + 1$, which is even.

1.5.1 $d_0 - d_1 = c_0 - c_1 = \textit{ even }.$ Then, given the position of e_{1_s} and e_{1_t}, by case 1 of lemma 1.1, we can construct the Hamilton Path in the way we want. Then connect e_{1_t} and e_{2_s} to connect those two rectangles.

1.5.2 $d_0 - d_1 = c_0 - c_1 = \textit{ odd and } h_0 = \textit{ odd }.$ Then, the two subgraph G_8 and G_9 will be the rectangle with odd weight and odd height